1 The Real Number System 1.5

The set of real numbers is the set of all numbers on the number line. For every point on the number line there exists a real number and for ever real number there exists a point on the number line. By the end of this section, you will be able to solve the following four problems:

1. Simplify: $2^2 + [5^3 - 3(4^2 - 5)]$

2. Graph the numbers on the number line: 4.5, -0.25, $\frac{3}{8}$, -$\frac{6}{3}$, $3\frac{1}{3}$, 0, 2, -3.

3. Use the associative property to fill in the blanks in the statement: $\_\_ \times (5 \times 7) = (6 \times \_\_) \times 7$

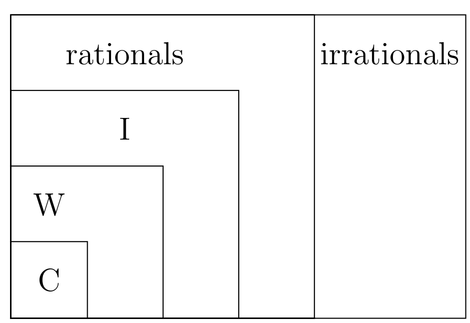
4. Use the commutative property to complete the statement: $5 \times (-6) = (\quad) \times 5$

2 Concepts

The real numbers are composed of a hierarchy of sets all of which can be assigned a position, either approximate or exact, on a number line. The counting numbers can be written as a set in the following way: we let C = $\{1, 2, 3, \cdots\}$. The whole numbers are the counting numbers plus zero. We let

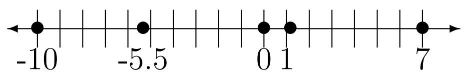
W=$\{0, 1, 2, 3, \cdots\}$. The integers, I=$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$, are the whole numbers plus the negatives of the counting numbers. The rational number are formed by constructing ratios from integers. Some examples are: $-\frac{2}{3}, 0.22, 2\frac{1}{2}, 3$. Notice that the number 3 is a rational number because it can be written as $\frac{3}{1}$. The final set that makes up the set of real numbers is the set of irrational numbers. These numbers cannot be written as the ratio of integers and their decimal expansions neither terminate nor repeat. Some examples are $-\sqrt{5}, \sqrt{2}, \pi$. To show how these number relate to each other we have constructed a Venn diagram of the relationships.

The Real Numbers



Notice: the real numbers = the rationals + the irrationals.

An example of a graph of real numbers on a number line is shown below:



On this number line, the arrow tips indicate that the line extends infinitely in both directions, and it is a convention that the hatch marks on the line be equally spaced.

2.1 Axioms of the Real Number System

The reason we discuss the axioms of the real line at all is because we will use them again and again to solve equations.

For our purposes, we treat a real number as an undefined entity and treat commutation, associativity, identity and inverse operations as axioms.

2.2 Commutation

The first property of the real numbers we investigate is called the commutative property. It is clear that if we add two real numbers, the order in which we do it makes no difference. For example, 3+2=2+3. This is an example of the commutative property under addition. In general, we say that if x and y are any two real numbers, then $x + y = y + x$.

The same is true for multiplication. For example, $3 \times 2 = 2 \times 3$. This is an example of the commutative property under multiplication. In general, we say that if x and y are any two real numbers, then $x \times y = y \times x$.

2.3 Associativity

Another important property is associativity. An example under addition is (3+2)+1=3+(2+1). Parentheses indicate the operation is to be performed first. This property demonstrates that how we group numbers to be added does not effect the result. In General, we say that for any real numbers x, y, z, then $(x + y) + z = x + (y + z)$.

Multiplication is also associative. How we group numbers in multiplication does not effect the result. For example, $(3 \times 2) \times 1 = 3 \times (2 \times 1)$. In General, we say that for any real numbers x, y, z, $(x \times y) \times z = x \times (y \times z)$.

2.4 Identity Elements

Fundamental to the operations of addition and multiplication are the identity elements for these operations. Zero is the identity element under addition.

For example, 2+0=2. In general, we say that if x is a real number, then $x + 0 = x$. Under multiplication, the identity element is 1 because multiplication by 1 does nothing. For example, $2 \times 1 = 2$, and in general we say for any real number x, $x \times 1 = x$

The identity element is essential for performing arithmetic on real numbers, but so is a way of getting back to the identity element. This is done by inverse operations. To return to zero under addition, we add the negative of a real number to itself. For example, 2+(-2)=0. This is the inverse property in action. In general, we say that for any real number x there exist a real number $-x$ such that, $x + (-x) = 0$.

Under multiplication, the way we return to the identity element is by multiplying by the inverse of a number. For example, the inverse of the number 3 is $\frac{1}{3}$, and if we multiply them together, we get 1: $3 \times \frac{1}{3} = 1$. In general, we say for any real number x, there exists a number $\frac{1}{x}$ $x \neq 0$ such that $x \times \frac{1}{x} = 1$.

Another property of the real numbers under addition and multiplication is *Closure*. All this property states is that if I add any two real numbers, I always get a real number, and if I multiply any two real numbers together, I always get a real number. In notation we write: for any two real numbers a

and b,

$$a + b = c \ is \ real$$

and

$$a \times b \ is \ real$$

2.5 Distributive Law

The last axiom we state is often called the distributive law which says that for any three real numbers, where a sum is multiplied by a number, we may add first and then multiply or multiply first and then add. For example, 3(1 + 2)= 3(3)=9 or $3 \times 1 + 3 \times 2 = 3 + 6 = 9$. In general, we say that for any three real numbers x, y, z that $x(y + z) = x \times y + x \times z$

We state the summary of these axioms in pairs for all real number x, y, z.

	Addition	Multiplication
Commutativity	x+y=y+x	$x \times y = y \times x$
Associativity	(x+y)+z=x+(y+z)	$(x \times y) \times z = x \times (y \times z)$
Identity	x+0=x	$x \times 1 = x$
Inverses	x + (-x)=0	$x \times \frac{1}{x} = 1 \quad x \neq 0$
Distributive law	$x(y + z) = x \times y + x \times z$	

3 The Order of Operations

How do we decide if $2 \times 3 + 4 = 10$ or if $2 \times 3 + 4 = 14$. All arithmetic operations are binary. The problem is that 3 cannot belong to *plus* and *times* at the same time. This problem is settled by an agreement called the order of operation. You may have heard the mnemonic, "Please Excuse My Dear Aunt Sally". Here we extent the mnemonic to read, "Please Excuse My Dear Aunt Sally Left to Right and Inside Out". There are five parts to evaluating expressions.

1. **Please**—perform operations in **Parentheses** first.
2. **Excuse**—raise numbers to their **Exponents** second.
3. **My Dear**—perform **Multiplication** and **Division** in order from **Left** to **Right**.
4. **Aunt Sally**—perform **Addition** and **Substraction** in order from **Left** to **Right**.
5. **Inside Out**—-If there are nested expressions with parentheses, brackets, and curly braces these should be performed from **Inside Out**

For example, we perform the order of operations on the expression: $5^2 + 3\{4 - 2[-1(11 - 4)]\} - (2^2 + 4)$ in the following way.

$$5^2 + 3\{4 - 2[-1(11 - 4)]\} - (2^2 + 4) \textit{ problem restatement}$$

$$25 + 3\{4 - 2[-1(11 - 4)]\} - (4 + 4) \textit{ evaluate exponents}$$

$$25 + 3\{4 - 2[-1(7)]\} - (8) \textit{ perform operations in parentheses}$$

$$25 + 3\{4 - 2[-7]\} - (8) \textit{ perform operations inside square brackets}$$

$$25 + 3\{4 + 14\} - (8) \textit{ perform operations inside curly braces}$$

$$25 + 3\{18\} - (8) \textit{ multiply}$$

$$25 + 54 - (8) \textit{ add}$$

$$79 - 8 \; subtract$$

$$71$$

4 Facts

1. For every real number there exists a point on the number line and for every point on the number line there exits a real number.

2. Every real number has an additive inverse. That is, for any x, there exists $-x$ such that $x + (-x) = 0$.

3. Every real number (except zero) has a multiplicative inverse. That is, for any x, there exists $\frac{1}{x}$ such that $x \times \frac{1}{x} = 1$

4. The commutative property states that the order in which we add two numbers or multiply two numbers makes no difference.

5. The Associative property states that the order in which we group two numbers under addition or multiplication makes no difference.

6. The extended mnemonic for The Order of Operations is: **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally **L**eft to **R**ight and **I**nside **O**ut.

5 Exercises

1. Simplify: $2^2 + [5^3 - 3(4^2 - 5)]$

2. Graph the numbers on the number line: 4.5, -0.25, $\frac{3}{8}$, -$\frac{6}{3}$, $3\frac{1}{3}$, 0, 2, -3.

3. Use the associative property to fill in the blanks in the statement: $\_\_ \times (5 \times 7) = (6 \times \_\_) \times 7$

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6 Solutions

1.

$$2^2 + [5^3 - 3(4^2 - 5)]$$

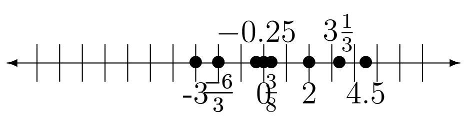
$$4 + [125 - 3(16 - 5)]$$

$$4 + [125 - 3(11)]$$

$$4 + [125 - 33]$$

$$4 + 92$$

$$96$$

2. 

3. $6 \times (5 \times 7) = (6 \times 5) \times 7$

4. $5 \times (-6) = (-6) \times 5$